11. cvičení - řešení

Příklad 1 (a)
$\frac{1}{\sin x}$ je definováno pro $x \neq k\pi, k \in \mathbb{Z}$. Má smysl tedy integrovat jen na intervalech $(k\pi, (k+1)\pi), k \in \mathbb{Z}$.

$$\int \frac{1}{\sin x} \mathrm{d}x = |y = \cos x, \mathrm{d}y = -\sin x \mathrm{d}x| = \int \frac{-\sin x}{-\sin^2 x} \mathrm{d}x = \int \frac{-\sin x}{-(1-\cos^2 x)} \mathrm{d}x =$$
$$= \int \frac{1}{y^2 - 1} \mathrm{d}y$$

Parciální zlomky: $\frac{1}{y^2-1} = \frac{A}{y-1} + \frac{B}{y+1} \implies 1 = Ay + A + By - B \implies A + B = 0, A - B = 1$

$$\int \frac{1}{y^2-1} \mathrm{d}y \stackrel{\text{lin.}}{=} -\frac{1}{2} \int \frac{1}{y+1} \mathrm{d}y + \frac{1}{2} \int \frac{1}{y-1} \mathrm{d}y \stackrel{\text{c}}{=} -\frac{1}{2} \log|y+1| + \frac{1}{2} \log|y-1|$$

$$\implies \int \frac{1}{\sin x} \mathrm{d}x \stackrel{\text{c}}{=} \frac{1}{2} \log|\cos x - 1| - \frac{1}{2} \log|\cos x + 1|$$

Příklad 1 (b)
Zřejmě je vnitřek integrálu definován pro všechna $x \in \mathbb{R}$.
Platí následující:

$$\tan^2 x = \frac{\sin^2 x}{\cos^2 x}$$
$$\tan^2 x + 1 = \frac{1}{\cos^2 x}$$
$$\cos^2 x = \frac{1}{\tan^2 x + 1}$$
$$\sin^2 x = 1 - \cos^2 x = 1 - \frac{1}{\tan^2 x + 1} = \frac{\tan^2 x + 1 - 1}{\tan^2 x + 1} = \frac{\tan^2 x}{\tan^2 x + 1}$$

$$\int \frac{1}{\sin^4 x + \cos^4 x} \mathrm{d}x = \left| y = \tan x, \mathrm{d}y = \frac{1}{\cos^2 x} \mathrm{d}x \right| = \int \frac{\cos^2 x}{\sin^4 x + \cos^4 x} \cdot \frac{1}{\cos^2 x} \mathrm{d}x =$$
$$= \int \frac{\frac{1}{y^2+1}}{\left(\frac{y^2}{y^2+1}\right)^2 + \left(\frac{1}{y^2+1}\right)^2} \mathrm{d}y = \int \frac{1}{y^2+1} \cdot \frac{(y^2+1)^2}{y^4+1} \mathrm{d}y = \int \frac{y^2+1}{y^4+1} \mathrm{d}y$$

Rozklad $y^4 + 1$ pomocí finty z reciprokých rovnic.
Vizte: https://www2.karlin.mff.cuni.cz/˜ portal/komplexni_cisla.dp/?page=rovnice-reciproke

$$y^4 + 1 = y^2 \left(y^2 + \frac{1}{y^2}\right)$$
$$z = y + \frac{1}{y} \implies z^2 = y^2 + 2 + \frac{1}{y^2}$$
$$\implies y^2 + \frac{1}{y^2} = z^2 - 2 = \left(z - \sqrt{2}\right)\left(z + \sqrt{2}\right)$$
$$\implies y^4 + 1 = y^2(y^2 + \frac{1}{y^2}) = y^2 \left(z - \sqrt{2}\right)\left(z + \sqrt{2}\right) = y\left(y + \frac{1}{y} - \sqrt{2}\right) y \left(y + \frac{1}{y} + \sqrt{2}\right) =$$
$$= \left(y^2 - y\sqrt{2} + 1\right)\left(y^2 + y\sqrt{2} + 1\right)$$

Trojčleny $y^2 \pm y\sqrt{2}+1$ již nelze rozložit (záporný diskriminant).
Paricální zlomky:

$$
\begin{aligned}
&\frac{y^2+1}{\left(y^2-y\sqrt{2}+1\right)\left(y^2+y\sqrt{2}+1\right)} = \frac{Ay+B}{\left(y^2+y\sqrt{2}+1\right)} + \frac{Cy+D}{\left(y^2-y\sqrt{2}+1\right)}\\
&y^2+1 = (Ay+B)\left(y^2-y\sqrt{2}+1\right) + (Cy+D)\left(y^2+y\sqrt{2}+1\right)\\
&y^2+1 = Ay^3 - Ay^2\sqrt{2} + Ay + By^2 - By\sqrt{2} + B + Cy^3 + Cy^2\sqrt{2} + Cy + Dy^2 + Dy\sqrt{2} + D\\
&y^2+1 = y^3(A+C) + y^2\left(-A\sqrt{2}+B+C\sqrt{2}+D\right) + y\left(A-B\sqrt{2}+C+D\sqrt{2}\right) + B + D
\end{aligned}
$$

Porovnáním koeficientů dostáváme:

$$
\begin{aligned}
&y^3 \implies 0 = A+C \implies -A = C \overset{2}{\implies} A = 0\\
&y^2 \implies 1 = -A\sqrt{2}+B+C\sqrt{2}+D \overset{1}{\implies} 1 = 2C\sqrt{2}+B+D \overset{4}{\implies} C = 0\\
&y \implies 0 = A-B\sqrt{2}+C+D\sqrt{2} \overset{1+2}{\implies} 0 = \sqrt{2}(D-B) \implies B = D\\
&y^0 \implies 1 = B+D \overset{4}{\implies} B = D = \frac{1}{2}
\end{aligned}
$$

Máme tedy: $A=C=0, B=D=\frac{1}{2}$.

$$
\begin{aligned}
&\int \frac{y^2+1}{y^4+1}\mathrm{d}y \overset{\text{lin.}}{=} \frac{1}{2}\int \frac{1}{y^2+y\sqrt{2}+1}\mathrm{d}y + \frac{1}{2}\int \frac{1}{y^2-y\sqrt{2}+1}\mathrm{d}y =\\
&= \frac{1}{2}\int \frac{1}{\left(y+\frac{\sqrt{2}}{2}\right)^2+\frac{1}{2}}\mathrm{d}y + \frac{1}{2}\int \frac{1}{\left(y-\frac{\sqrt{2}}{2}\right)^2+\frac{1}{2}}\mathrm{d}y =\\
&= \int \frac{1}{\left(\frac{y+\frac{\sqrt{2}}{2}}{\frac{1}{\sqrt{2}}}\right)^2+1}\mathrm{d}y + \int \frac{1}{\left(\frac{y-\frac{\sqrt{2}}{2}}{\frac{1}{\sqrt{2}}}\right)^2+1}\mathrm{d}y =\\
&= \left| z = \sqrt{2}\left(y+\frac{\sqrt{2}}{2}\right), \mathrm{d}z = \sqrt{2}\mathrm{d}y, u = \sqrt{2}\left(y-\frac{\sqrt{2}}{2}\right), \mathrm{d}u = \sqrt{2}\mathrm{d}y \right| =\\
&= \frac{1}{\sqrt{2}}\int \frac{1}{z^2+1}\mathrm{d}z + \frac{1}{\sqrt{2}}\int \frac{1}{u^2+1}\mathrm{d}u \overset{\text{c}}{=} \frac{1}{\sqrt{2}}\arctan z + \frac{1}{\sqrt{2}}\arctan u =\\
&= \frac{1}{\sqrt{2}}\arctan \sqrt{2}\left(y+\frac{\sqrt{2}}{2}\right) + \frac{1}{\sqrt{2}}\arctan \sqrt{2}\left(y-\frac{\sqrt{2}}{2}\right) =\\
&= \frac{1}{\sqrt{2}}\arctan \sqrt{2}\left(\tan x+\frac{\sqrt{2}}{2}\right) + \frac{1}{\sqrt{2}}\arctan \sqrt{2}\left(\tan x-\frac{\sqrt{2}}{2}\right) =\\
&= \frac{1}{\sqrt{2}}\arctan\left(\sqrt{2}\tan x+1\right) + \frac{1}{\sqrt{2}}\arctan\left(\sqrt{2}\tan x-1\right)
\end{aligned}
$$

Příklad 1 (c) $\int \frac{\cos^3 x}{2-\sin x}\mathrm{d}x$
Podmínky: $\sin x \neq 2$ - platí vždy.
Z návodu: $R(x,y) = \frac{y^3}{1-x}$. Pak chceme spočítat $\int R(\sin x, \cos x)\mathrm{d}x$. Platí, že $R(\sin x, -\cos x) = -R(\sin x, \cos x)$. Budeme tedy volit substituci $y = \sin x$.

$$\int \frac{\cos^3 x}{2-\sin x}\mathrm{d}x = |y=\sin x, \mathrm{d}y = \cos x \mathrm{d}x| = \int \frac{(1-\sin^2 x)\cos x}{2-\sin x}\mathrm{d}x = \int \frac{1-y^2}{2-y}\mathrm{d}y =$$
$$\int \frac{y^2-1}{y-2}\mathrm{d}y = \int y+2+\frac{3}{y-2}\mathrm{d}y = |z=y-2, \mathrm{d}z=\mathrm{d}y| \overset{\text{lin.}}{=} \frac{y^2}{2}+2y+3\int \frac{1}{z}\mathrm{d}z =$$
$$\overset{\text{c}}{=} \frac{y^2}{2}+2y+3\log|z| = \frac{y^2}{2}+2y+3\log|y-2| = \frac{\sin^2 x}{2}+2\sin x+3\log|\sin x-2|$$

Příklad 1 (d) $\int \frac{1}{2-\cos x}\mathrm{d}x$

Všimněmě si, že $\frac{1}{2-\cos x}$ je definováno všude.

Dále, pokud $R(x,y) = \frac{1}{2-y}$, tak počítáme $\int \mathbb{R}(\sin x, \cos x)\mathrm{d}x$. Přičemž neplatí ani jedno z následujícího: $R(-\sin x, \cos x) = -R(\sin x, \cos x), R(\sin x, -\cos x) = -R(\sin x, \cos x), R(-\sin x, -\cos x) = R(\sin x, \cos x)$. Proto (dle návodu z teorie) provedeme substituci: $t = \tan\frac{x}{2}$. Odvoďme si vztahy pro $\sin x, \cos x, \mathrm{d}x$ vzhledem k této substituci:

$$t = \tan\frac{x}{2}$$
$$\tan^2\frac{x}{2} = \frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}} \implies 1+\tan^2\frac{x}{2} = 1+\frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}} = \frac{1}{\cos^2\frac{x}{2}} \qquad \text{(Pythagorova věta)}$$
$$\implies \cos^2\frac{x}{2} = \frac{1}{1+\tan^2\frac{x}{2}}$$
$$\mathrm{d}t = \frac{1}{\cos^2\frac{x}{2}}\cdot\frac{1}{2}\mathrm{d}x = \frac{1}{2}\left(1+\tan^2\frac{x}{2}\right)\mathrm{d}x = \frac{1}{2}\left(1+t^2\right)\mathrm{d}x \implies \frac{2}{1+t^2}\mathrm{d}t = \mathrm{d}x$$
$$\sin x = \sin\left(2\frac{x}{2}\right) = 2\sin\frac{x}{2}\cos\frac{x}{2} = 2\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\cos^2\frac{x}{2} = 2\tan\frac{x}{2}\cdot\frac{1}{1+\tan^2\frac{x}{2}} = \frac{2t}{1+t^2}$$
$$\cos x = \cos\left(2\frac{x}{2}\right) = \cos^2\frac{x}{2}-\sin^2\frac{x}{2} = \cos^2\frac{x}{2}-\left(1-\cos^2\frac{x}{2}\right) = \frac{1}{1+t^2}-\left(1-\frac{1}{1+t^2}\right) =$$
$$= \frac{1}{1+t^2}-\frac{1+t^2-1}{1+t^2} = \frac{1-t^2}{1+t^2}$$

Použijme tedy tuto substituci při výpočetu zadaného integrálu.

$$\int \frac{1}{2-\cos x}\mathrm{d}x = \left|t=\tan\frac{x}{2}, \frac{2}{1+t^2}\mathrm{d}t = \mathrm{d}x\right| = \int \frac{1}{2-\frac{1-t^2}{1+t^2}}\frac{2}{1+t^2}\mathrm{d}t == \int \frac{1}{\frac{2+2t^2-1+t^2}{1+t^2}}\frac{2}{1+t^2}\mathrm{d}t =$$
$$= \int \frac{1+t^2}{1+3t^2}\frac{2}{1+t^2}\mathrm{d}t = \int \frac{2}{(\sqrt{3}t)^2+1}\mathrm{d}t = \left|y=\sqrt{3}t, \mathrm{d}y = \sqrt{3}\mathrm{d}t\right| \overset{\text{lin.}}{=} \frac{2}{\sqrt{3}}\int \frac{1}{y^2+1}\mathrm{d}y \overset{\text{c}}{=} \frac{2}{\sqrt{3}}\arctan y =$$
$$= \frac{2}{\sqrt{3}}\arctan\left(\sqrt{3}t\right) = \frac{2}{\sqrt{3}}\arctan\left(\sqrt{3}\tan\frac{x}{2}\right)$$

Příklad 1 (e) $\int \frac{1}{1+\sin^2 x}\mathrm{d}x$

Opět $\frac{1}{1+\sin^2 x}$ je definováno všude.

Pro racionální funkci $R(x,y) = \frac{1}{1+x^2}$ platí: $\int \frac{1}{1+\sin^2 x}\mathrm{d}x = \int R(\sin x, \cos x)\mathrm{d}x$. Navíc platí: $R(-\sin x, -\cos x) = R(\sin x, \cos x)$. Volme tedy substituci $t = \tan x$.

Podobně jako výše se odvodí, že $\mathrm{d}x = \frac{1}{1+t^2}\mathrm{d}t, \sin^2 x = \frac{t^2}{1+t^2}, \cos^2 x = \frac{1}{1+t^2}$

$$\int \frac{1}{1+\sin^2 x}\mathrm{d}x = \left| t=\tan x, \mathrm{d}x = \frac{1}{1+t^2}\mathrm{d}t \right| = \int \frac{1}{1+\frac{t^2}{1+t^2}} \cdot \frac{1}{1+t^2}\mathrm{d}t = \int \frac{t^2+1}{1+t^2+t^2}\frac{1}{1+t^2}\mathrm{d}t =$$
$$= \frac{1}{1+2t^2}\mathrm{d}t = \int \frac{1}{1+\left(t\sqrt{2}\right)^2}\mathrm{d}t = \left| y = t\sqrt{2}, \mathrm{d}y = \sqrt{2}\mathrm{d}t \right| \overset{\text{lin.}}{=} \frac{1}{\sqrt{2}}\int \frac{1}{1+y^2}\mathrm{d}y \overset{\text{c}}{=} \frac{1}{\sqrt{2}}\arctan y =$$
$$= \frac{1}{\sqrt{2}}\arctan(\sqrt{2}t) = \frac{1}{\sqrt{2}}\arctan\left(\sqrt{2}\tan x\right)$$

Příklad 1 (f) $\int \frac{1}{\cos x \cdot \sin^3 x}\mathrm{d}x$

Podmínky: $\cos x \neq 0 \wedge \sin x \neq 0 \implies x \neq k\frac{\pi}{2}, k \in \mathbb{Z}$.

Pro $R(\sin x, \cos x) = \frac{1}{\cos x \cdot \sin^3 x}$ platí: $R(-\sin x, \cos x) = -R(\sin x, \cos x)$. Volme tedy substituci: $t = \cos x$.

$$\int \frac{1}{\cos x \cdot \sin^3 x}\mathrm{d}x = |t = \cos x, \mathrm{d}t = -\sin x \mathrm{d}x| = \int \frac{-\sin x}{-\cos x \cdot \sin^4 x}\mathrm{d}x = \int \frac{-\sin x}{-\cos x \cdot (1-\cos^2 x)^2}\mathrm{d}x =$$
$$= \int \frac{-1}{t \cdot (1-t^2)^2}\mathrm{d}t$$

Parciální zlomky:

$$\frac{-1}{t(t^2-1)^2} = \frac{-1}{t(t+1)^2(t-1)^2} = \frac{A}{t} + \frac{B}{t+1} + \frac{C}{(t+1)^2} + \frac{D}{t-1} + \frac{E}{(t-1)^2}$$
$$-1 = A(t^2-1)^2 + Bt(t+1)(t-1)^2 + Ct(t-1)^2 + Dt(t-1)(t+1)^2 + Et(t+1)^2$$
$$-1 = A(t^4-2t^2+1) + (t-1)^2(Bt^2+Bt+Ct) + (t+1)^2(Dt^2-Dt+Et)$$
$$-1 = At^4 - 2At^2 + A + (t^2-2t+1)(Bt^2+(B+C)t) + (t^2+2t+1)(Dt^2+(E-D)t)$$
$$-1 = At^4 - 2At^2 + A + Bt^4 + (B+C)t^3 - 2Bt^3 - 2(B+C)t^2 + Bt^2 + (B+C)t + Dt^4 +$$
$$+ (E-D)t^3 + 2Dt^3 + 2(E-D)t^2 + Dt^2 + (E-D)t$$
$$-1 = t^4(A+B+D) + t^3(B+C-2B+E-D+2D) +$$
$$+ t^2(-2A-2B-2C+B+2E-2D+D) + t(B+C+E-D) + A$$

Porovnáním koeficientů dostátáme soustavu:

$$A+B+D=0 \overset{5}{\implies} B+D=1 \overset{4}{\implies} B=D=\frac{1}{2}$$
$$-B+C+D+E=0 \implies E+C=B-D$$
$$-2A-B-2C-D+2E=0 \overset{1+5+4}{\implies} 2-\frac{1}{2}-2C-\frac{1}{2}-2C=0 \implies -4C=-1 \implies C=\frac{1}{4}$$
$$B+C+E-D=0 \implies E+C=D-B \overset{2}{\implies} D-B=B-D \implies D=B \implies E=-C$$
$$-1=A$$

Řešení soustavy je: $A=-1, B=\frac{1}{2}, C=\frac{1}{4}, D=\frac{1}{2}, E=-\frac{1}{4}$

$$\int \frac{-1}{t(1-t^2)^2}\mathrm{d}t = \int -\frac{1}{t} + \frac{1}{2}\frac{1}{t+1} + \frac{1}{4}\frac{1}{(t+1)^2} + \frac{1}{2}\frac{1}{t-1} - \frac{1}{4}\frac{1}{(t-1)^2}\mathrm{d}t = |\text{substituce}| =$$
$$\overset{c}{=} -\log|t| + \frac{1}{2}\log|t+1| - \frac{1}{4}\frac{1}{t+1} + \frac{1}{2}\log|t-1| + \frac{1}{4}\frac{1}{t-1} =$$
$$= \frac{1}{4}\frac{t+1-(t-1)}{(t-1)(t+1)} + \frac{1}{2}\log|(t+1)(t-1)| - \log|t| = \frac{1}{4}\frac{2}{t^2-1} + \frac{1}{2}\left(\log|t^2-1| - \log|t^2|\right) =$$
$$= \frac{1}{2(t^2-1)} + \frac{1}{2}\log\left|\frac{t^2-1}{t^2}\right| = \frac{1}{2(\cos^2 x)-1} + \log\sqrt{\frac{1-\cos^2 x}{\cos^2 x}} = -\frac{1}{2\sin^2 x} + \log\sqrt{\frac{\sin^2 x}{\cos^2 x}} =$$
$$= -\frac{1}{2\sin^2 x} + \log|\tan x|$$

Příklad 1 (g) $\int \frac{\sin x}{1+\sin x}\mathrm{d}x$

Podmínky: $\sin x \neq -1 \implies x \neq \frac{3\pi}{2} + k2\pi, k \in \mathbb{Z}$.

Pro $R(\sin x, \cos x) = \frac{\sin x}{1+\sin x}$ neplatí ani jedno z následujícího: $R(-\sin x, \cos x) = -R(\sin x, \cos x), R(\sin x, -\cos x) = -R(\sin x, \cos x), R(-\sin x, -\cos x) = R(\sin x, \cos x)$. Proto (dle návodu z teorie) provedeme substituci: $t = \tan\frac{x}{2}$.

$$\int \frac{\sin x}{1+\sin x}\mathrm{d}x = \left|t = \tan\frac{x}{2}, \mathrm{d}x = \frac{2}{1+t^2}\mathrm{d}t\right| = \int \frac{\frac{2t}{1+t^2}}{1+\frac{2t}{1+t^2}}\frac{2}{1+t^2}\mathrm{d}t = \int \frac{4t}{(1+t^2)^2}\cdot\frac{1+t^2}{1+t^2+2t}\mathrm{d}t =$$
$$= \int \frac{4t}{(t^2+1)(t+1)^2}\mathrm{d}t$$

Parciální zlomky:

$$\frac{4t}{(t^2+1)(t+1)^2} = \frac{At+B}{t^2+1} + \frac{C}{t+1} + \frac{D}{(t+1)^2}$$
$$4t = (At+B)(t^2+2t+1) + C(t+1)(t^2+1) + D(t^2+1)$$
$$4t = At^3 + 2At^2 + At + Bt^2 + 2Bt + B + Ct^3 + Ct + Ct^2 + C + Dt^2 + D$$

Porovnáním koeficientů dostátáme soustavu:

$$0 = A + C \overset{2}{\implies} C = 0$$
$$0 = 2A + B + C + D \overset{4}{\implies} 0 = 2A \implies A = 0$$
$$4 = A + 2B + C \overset{1}{\implies} 4 = 2B \implies B = 2$$
$$0 = B + C + D \implies 0 = 2 + 0 + D \implies D = -2$$

$$\int \frac{4t}{(t^2+1)(t+1)^2}\mathrm{d}t \overset{\text{lin.}}{=} \int \frac{2}{t^2+1}\mathrm{d}t + \int \frac{-2}{(t+1)^2}\mathrm{d}t = |y = t+1, \mathrm{d}y = \mathrm{d}t| = 2\arctan t - 2\int \frac{1}{y^2}\mathrm{d}y =$$
$$\overset{c}{=} 2\arctan t + 2\frac{1}{y} = 2\arctan t + 2\frac{1}{t+1} = 2\arctan\left(\tan\frac{x}{2}\right) + 2\frac{1}{\tan\frac{x}{2}+1} = 2\frac{x}{2} + \frac{2}{\tan\frac{x}{2}+1} =$$
$$= x + \frac{2}{\tan\frac{x}{2}+1}$$

Příklad 1 (h) $\int \frac{2-\sin x}{2+\cos x}\mathrm{d}x$

$\frac{2-\sin x}{2+\cos x}$ je definováno všude.

$$\int \frac{2-\sin x}{2+\cos x}\mathrm{d}x = \left|t=\tan\frac{x}{2}, \mathrm{d}x = \frac{2}{1+t^2}\mathrm{d}t\right| = \int \frac{2-\frac{2t}{1+t^2}}{2+\frac{1-t^2}{1+t^2}}\frac{2}{1+t^2}\mathrm{d}t = \int \frac{\frac{2+2t^2-2t}{1+t^2}}{\frac{2+2t^2+1-t^2}{1+t^2}}\frac{2}{1+t^2}\mathrm{d}t =$$
$$= \int \frac{2t^2-2t+2}{1+t^2}\cdot\frac{1+t^2}{t^2+3}\cdot\frac{2}{1+t^2}\mathrm{d}t = \int \frac{2(2t^2-2t+2)}{(t^2+1)(t^2+3)}\mathrm{d}t$$

Parciální zlomky:

$$\frac{4t^2-4t+4}{(t^2+1)(t^2+3)} = \frac{At+B}{t^2+1}+\frac{Ct+D}{t^2+3}$$
$$4t^2-4t+4 = (At+B)(t^2+3)+(Ct+D)(t^2+1)$$
$$4t^2-4t+4 = At^3+3At+Bt^2+3B+Ct^3+Ct+Dt^2+D$$

Porovnáním koeficientů dostáváme:

$$0 = A + C \implies C = -A \stackrel{3}{\implies} C = 2$$
$$4 = B + D \stackrel{4}{\implies} D = 4$$
$$-4 = 3A + C \stackrel{1}{\implies} -4 = 2A \implies A = -2$$
$$4 = 3B + D \stackrel{2}{\implies} 0 = 2B \implies B = 0$$

$$\int \frac{4t^2-4t+4}{(t^2+1)(t^2+3)}\mathrm{d}t \stackrel{\text{lin.}}{=} \int \frac{-2t}{t^2+1}\mathrm{d}t + \int \frac{2t+4}{t^2+3}\mathrm{d}t = \left|y=t^2+1, \mathrm{d}y = 2t\mathrm{d}t, z = t^2+3, \mathrm{d}z = 2t\mathrm{d}t\right| =$$
$$= -\int \frac{1}{y}\mathrm{d}y + \int \frac{1}{z}\mathrm{d}z + 4\int \frac{1}{t^2+3}\mathrm{d}t = \left|u = \frac{t}{\sqrt{3}}, \mathrm{d}u = \frac{1}{\sqrt{3}}\mathrm{d}t\right| = -\log|y| + \log|z| + 4\sqrt{3}\int \frac{1}{u^2+1}\mathrm{d}u =$$
$$\stackrel{c}{=} -\log|t^2+1| + \log|t^2+3| + 4\sqrt{3}\arctan u = \log\frac{t^2+3}{t^2+1} + 4\sqrt{3}\arctan\frac{t}{\sqrt{3}} =$$
$$= \log\frac{\tan^2\frac{x}{2}+3}{\tan^2\frac{x}{2}+1} + 4\sqrt{3}\arctan\frac{\tan\frac{x}{2}}{\sqrt{3}}$$

Příklad 1 (i) $\int \frac{\sin^3 x}{1+4\cos^2 x+3\sin^2 x}\mathrm{d}x$

Vnitřek integrálu je definován všude. Jelikož pro $R(x,y) = \frac{x^3}{1+4y^2+3x^2}$ platí, že $R(-x,y) = -R(x,y)$, tak volíme substituci: $t = \cos x$.

Z Pythagorovy věty plyne první rovnost níže.

$$\int \frac{\sin^3 x}{1+4\cos^2 x+3\sin^2 x}\mathrm{d}x = \int \frac{\sin^3 x}{4+\cos^2 x}\mathrm{d}x = |t=\cos x, \mathrm{d}t = -\sin x\mathrm{d}x| = \int \frac{-(1-t^2)}{4+t^2}\mathrm{d}t =$$
$$= \int \frac{t^2-1}{t^2+4}\mathrm{d}t = \int \frac{t^2-1+5-5}{t^2+4}\mathrm{d}t \stackrel{\text{lin.}}{=} \int 1\mathrm{d}t - 5\int \frac{1}{t^2+4}\mathrm{d}t = t - \frac{5}{4}\int \frac{1}{\left(\frac{t}{2}\right)^2+1}\mathrm{d}t =$$
$$= \left|y = \frac{t}{2}, \mathrm{d}y = \frac{1}{2}\mathrm{d}t\right| = t - \frac{5}{4}\cdot 2\int \frac{1}{y^2+1}\mathrm{d}y \stackrel{c}{=} t - \frac{5}{2}\arctan y = t - \frac{5}{2}\arctan\frac{t}{2} =$$
$$= \cos x - \frac{5}{2}\arctan\frac{\cos x}{2}$$

Příklad 1 (j) $\int \frac{3\sin^2 x+\cos^2 x}{\sin^2 x+3\cos^2 x}\mathrm{d}x$
Vnitřek integrálu je zřejmě definován všude.

Pro $\mathbb{R}(\sin x, \cos x) = \frac{3\sin^2 x+\cos^2 x}{\sin^2 x+3\cos^2 x}$ platí, že $R(-\sin x, -\cos x) = R(\sin x, \cos x)$. Proto volíme substituci $t = \tan x$. Integrujeme tedy jen na intervalech $(-\frac{\pi}{2}+k\pi, \frac{\pi}{2}+k\pi), k \in \mathbb{Z}$.

Níže první rovnost plyne z Pythagorovy věty.

$$\int \frac{3\sin^2 x+\cos^2 x}{\sin^2 x+3\cos^2 x}\mathrm{d}x = \int \frac{2\sin^2 x+1}{1+2\cos^2 x}\mathrm{d}x = \left|t = \tan x, \mathrm{d}x = \frac{1}{t^2+1}\mathrm{d}t\right| = \int \frac{2\frac{t^2}{1+t^2}+1}{1+2\frac{1}{1+t^2}} \cdot \frac{1}{t^2+1}\mathrm{d}t =$$
$$= \int \frac{2t^2+1+t^2}{1+t^2} \cdot \frac{1+t^2}{1+t^2+2} \cdot \frac{1}{t^2+1}\mathrm{d}t = \int \frac{3t^2+1}{(t^2+3)(t^2+1)}\mathrm{d}t$$

Parciální zlomky:

$$\frac{3t^2+1}{(t^2+3)(t^2+1)} = \frac{At+B}{t^2+3} + \frac{Ct+D}{t^2+1}$$
$$3t^2+1 = (At+B)(t^2+1) + (Ct+D)(t^2+3)$$
$$3t^2+1 = At^3+At+Bt^2+B+Ct^3+3Ct+Dt^2+3D$$

Porovnáním koeficientů dostáváme:

$$0 = A + C$$
$$3 = B + D \stackrel{4}{\Longrightarrow} 3 = B - 1 \implies B = 4$$
$$0 = A + 3C \stackrel{1}{\Longrightarrow} 0 = 2C \implies C = 0 \stackrel{1}{\Longrightarrow} A = 0$$
$$1 = B + 3D \stackrel{2}{\Longrightarrow} 1 = 3 + 2D \implies 2D = -2 \implies D = -1$$

$$\int \frac{3t^2+1}{(t^2+3)(t^2+1)}\mathrm{d}t \stackrel{\text{lin.}}{=} 4\int \frac{1}{t^2+3}\mathrm{d}t - \int \frac{1}{t^2+1}\mathrm{d}t = \frac{4}{3}\int \frac{1}{\left(\frac{t}{\sqrt{3}}\right)^2+1}\mathrm{d}t - \arctan t =$$
$$= \left|u = \frac{t}{\sqrt{3}}, \mathrm{d}u = \frac{1}{\sqrt{3}}\mathrm{d}t\right| = \frac{4\sqrt{3}}{3}\int \frac{1}{y^2+1}\mathrm{d}y - \arctan t \stackrel{\text{c}}{=} \frac{4\sqrt{3}}{3}\arctan y - \arctan t =$$
$$= \frac{4\sqrt{3}}{3}\arctan \frac{t}{\sqrt{3}} - \arctan t = \frac{4\sqrt{3}}{3}\arctan \frac{\tan x}{\sqrt{3}} - \arctan \tan x = \frac{4\sqrt{3}}{3}\arctan \frac{\tan x}{\sqrt{3}} - x$$

Příklad 3 (a) $\int \frac{\log x}{x - x\log x}\mathrm{d}x$
Podmínky: $x > 0, x \neq 0, x \neq e$.

$$\int \frac{\log x}{x - x\log x}\mathrm{d}x = \int \frac{1}{x}\frac{\log x}{1-\log x}\mathrm{d}x = \left|y = \log x, \mathrm{d}y = \frac{1}{x}\mathrm{d}x\right| = \int \frac{y}{1-y}\mathrm{d}y =$$
$$= \int \frac{y-1+1}{1-y}\mathrm{d}y \stackrel{\text{lin.}}{=} \int -1\mathrm{d}y + \int \frac{-1}{y-1}\mathrm{d}y = |u = y-1, \mathrm{d}u = \mathrm{d}y| = -y - \int \frac{1}{u}\mathrm{d}u =$$
$$\stackrel{\text{c}}{=} -y - \log|u| = -y - \log|y-1| = -\log x - \log|\log x - 1|$$

Příklad 3 (b) $\int \frac{1}{e^{2x}+e^x-2}\mathrm{d}x$

Podmínky: $e^{2x} + e^x - 2 = (e^x - 1)(e^x + 2) \neq 0 \implies x \neq 0$.

$$\int \frac{1}{e^{2x} + e^x - 2} \mathrm{d}x = \int \frac{1}{(e^x)^2 + e^x - 2} \mathrm{d}x = |y = e^x, \mathrm{d}y = e^x \mathrm{d}x| = \int \frac{1}{y(y^2 + y - 2)} \mathrm{d}y$$

Parciální zlomky:

$$\begin{aligned}
&\frac{1}{y(y^2 + y - 2)} = \frac{1}{y(y+2)(y-1)} = \frac{A}{y} + \frac{B}{y+2} + \frac{C}{y-1} \\
&1 = A(y^2 + y - 2) + By(y-1) + Cy(y+2) \\
&1 = Ay^2 + Ay - 2A + By^2 - By + Cy^2 + 2Cy
\end{aligned}$$

Porovnáním koeficientů dostáváme:

$$\begin{aligned}
&0 = A + B + C \overset{3}{\implies} -B = C - \frac{1}{2} \overset{2}{\implies} B = -\frac{1}{3} + \frac{1}{2} = \frac{1}{6} \\
&0 = A - B + 2C \overset{1+3}{\implies} 0 = -\frac{1}{2} + C - \frac{1}{2} + 2C \implies 3C = 1 \implies C = \frac{1}{3} \\
&1 = -2A \implies A = -\frac{1}{2}
\end{aligned}$$

$$\int \frac{1}{y(y^2 + y - 2)} \mathrm{d}y \overset{\text{lin.}}{=} -\frac{1}{2} \int \frac{1}{y} \mathrm{d}y + \frac{1}{6} \int \frac{1}{y+2} \mathrm{d}y + \frac{1}{3} \int \frac{1}{y-1} \mathrm{d}y =$$
$$= -\frac{1}{6} \log|y| + \frac{1}{6} \log|y+2| + \frac{1}{3} \log|y-1| = -\frac{1}{2} \log(e^x) + \frac{1}{6} \log(e^x + 2) + \frac{1}{3} \log|e^x - 1| = -\frac{1}{2} x + \frac{1}{6} \log(e^x + 2) + \frac{1}{3} \mathrm{lo}$$

Poznámka: symbolem log máme na mysli přirozený logaritmus.

Příklad 3 (c) $\int \frac{\sqrt[4]{x}}{\sqrt[4]{x} + \sqrt{x}} \mathrm{d}x$
Podmínky: $x > 0$.

$$\int \frac{\sqrt[4]{x}}{\sqrt[4]{x} + \sqrt{x}} \mathrm{d}x = \left| y = \sqrt[4]{x}, \mathrm{d}y = \frac{1}{4} x^{-\frac{3}{4}} \mathrm{d}x \right| = \int \frac{y}{y + y^2} \cdot 4y^3 \mathrm{d}y \overset{\text{lin.}}{=} 4 \int \frac{y^3}{y+1} \mathrm{d}y =$$
$$= 4 \int y^2 - y + 1 - \frac{1}{y+1} \mathrm{d}y \overset{\text{c}}{=} \frac{4}{3} y^3 - 2y^2 + 4y - \log|y+1| = \frac{4}{3} \sqrt{x^3} - 2\sqrt{x} + 4\sqrt[4]{x} - \log|\sqrt[4]{x} + 1|$$

Příklad 3 (d) $\int \frac{\sqrt{2x+3} + x}{\sqrt{2x+3} - x} \mathrm{d}x$

Podmínky: $2x + 3 \geq 0 \implies x \geq -\frac{3}{2}$ a $2x + 3 - x^2 = 0 \implies x \neq 3, x \neq -1$. Vnitřek integrálu je tedy definován na $\left[-\frac{3}{2}, -1\right) \cup (-1, 3) \cup (0, \infty)$. Má tedy smyl integrál uvažovat na intervalech $\left(-\frac{3}{2}, -1\right), (-1, 3), (3, \infty)$.

$$\int \frac{\sqrt{2x+3} + x}{\sqrt{2x+3} - x} \mathrm{d}x = \left| y = \sqrt{2x+3}, \frac{y^2 - 3}{2} = x, \mathrm{d}y = \frac{1}{\sqrt{2x+3}} \mathrm{d}x \right| = \int \frac{y + \frac{y^2-3}{2}}{y - \frac{y^2-3}{2}} \cdot y \mathrm{d}y =$$
$$= \int \frac{2y + y^2 - 3}{2y - y^2 + 3} y \mathrm{d}y = \int \frac{-y^3 - 2y^2 + 3y}{y^2 - 2y - 3} \mathrm{d}y = \int -y - 4 + \frac{-8y - 12}{y^2 - 2y - 3} \mathrm{d}y$$

Paricální zlomky:

$$
\begin{aligned}
&\frac{-8y-12}{y^2-2y-3} = \frac{-8y-12}{(y-3)(y+1)} = \frac{A}{y-3} + \frac{B}{y+1} \\
&-8y-12 = A(y+1) + B(y-3) \\
&-8y-12 = Ay + A + By - 3B
\end{aligned}
$$

Porovnáním koeficientů dostáváme:

$$
\begin{aligned}
-8 &= A + B \\
-12 &= A - 3B
\end{aligned}
$$

Odečtením rovnic dostáváme $4 = 4B \implies B = 1, A = -8 - 1 = -9$.

$$
\begin{aligned}
&\int -y - 4 + \frac{-8y-12}{y^2-2y-3}\mathrm{d}y \overset{\text{lin.}}{=} -\frac{y^2}{2} - 4y - 9\int \frac{1}{y-3}\mathrm{d}y + \int \frac{1}{y+1}\mathrm{d}y = \\
&\overset{\text{c}}{=} -\frac{y^2}{2} - 4y - 9\log|y-3| + \log|y+1| = \\
&= -\frac{2x+3}{2} - 4\sqrt{2x+3} - 9\log|\sqrt{2x+3}-3| + \log\left(\sqrt{2x+3}+1\right)
\end{aligned}
$$

Příklad 3 (e) $\int \frac{e^{4x}+e^{2x}}{e^{3x}-1}\mathrm{d}x$
Podmínky: $e^{3x} \neq 1 \implies 3x \neq 0 \implies x \neq 0$

$$
\begin{aligned}
&\int \frac{e^{4x}+e^{2x}}{e^{3x}-1}\mathrm{d}x = |y = e^x, \mathrm{d}y = e^x\mathrm{d}x| = \int \frac{y^3+y}{y^3-1}\mathrm{d}y = \int \frac{y^3-1+y+1}{y^3-1}\mathrm{d}y = \\
&\overset{\text{lin.}}{=} y + \int \frac{y+1}{y^3-1}\mathrm{d}y
\end{aligned}
$$

Parciální zlomky:

$$
\begin{aligned}
&\frac{y+1}{y^3-1} = \frac{y+1}{(y-1)(y^2+y+1)} = \frac{A}{y-1} + \frac{By+C}{y^2+y+1} \\
&y+1 = Ay^2 + Ay + A + By^2 - By + Cy - C \\
&y+1 = y^2(A+B) + y(A-B+C) + A - C
\end{aligned}
$$

Porovnáním koeficientů dostáváme:

$$
\begin{aligned}
&0 = A + B \implies -B = A \\
&1 = A - B + C \overset{1+3}{\implies} 1 = A + A + A - 1 \implies 3A = 2 \implies A = \frac{2}{3} \\
&1 = A - C \implies C = A - 1
\end{aligned}
$$

Máme tedy: $A = \frac{2}{3}, B = -\frac{2}{3}, C = -\frac{1}{3}$.

$$y+\int\frac{y+1}{y^3-1}\mathrm{d}y = y+\frac{2}{3}\int\frac{1}{y-1}\mathrm{d}y-\frac{1}{3}\int\frac{2y+1}{y^2+y+1} = \left|u=y^2+y+1, \mathrm{d}u=(2y+1)\mathrm{d}y\right| =$$
$$= y+\frac{2}{3}\log|y-1|-\frac{1}{3}\int\frac{1}{u}\mathrm{d}u \overset{c}{=} y+\frac{2}{3}\log|y-1|-\frac{1}{3}\log|y^2+y+1| =$$
$$= e^x+\frac{2}{3}\log|e^x-1|-\frac{1}{3}\log(e^{2x}+e^x+1)$$

Příklad 3 (f) $\int\frac{1}{x\log x\cdot\log(\log x)}\mathrm{d}x$

$$\int\frac{1}{x\log x\cdot\log(\log x)}\mathrm{d}x = \left|y=\log x, \mathrm{d}y=\frac{1}{x}\mathrm{d}x\right| = \int\frac{1}{y\log(y)}\mathrm{d}y = \left|u=\log y, \mathrm{d}u=\frac{1}{y}\mathrm{d}y\right| = \int\frac{1}{u}\mathrm{d}u =$$
$$\overset{c}{=} \log|u| = \log|\log y| = \log|\log(\log x)|$$

Příklad 3 (g) $\int\frac{1}{x}\sqrt{\frac{x+2}{x-3}}\mathrm{d}x$

Provedeme substituci $t=\sqrt{\frac{x+2}{x-3}}$. Vyjádřeme nejdříve x v závislosti na t.

$$t^2=\frac{x+2}{x-3} \implies t^2(x-3)=x+2 \implies x(t^2-1)=2+3t^2 \implies x=\frac{3t^2+2}{t^2-1}$$

Zderivováním získaného vztahu dostaneme vztah pro dx a dt:

$$\mathrm{d}x=\frac{6t(t^2-1)-(3t^2+2)(2t)}{(t^2-1)^2}\mathrm{d}t=\frac{6t^3-6t-6t^3-4t}{(t^2-1)^2}\mathrm{d}t=\frac{-10t}{(t^2-1)^2}\mathrm{d}t$$

$$\int\frac{1}{x}\sqrt{\frac{x+2}{x-3}}\mathrm{d}x = \left|t=\sqrt{\frac{x+2}{x-3}}, x=\frac{3t^2+2}{t^2-1}, \mathrm{d}x=\frac{-10t}{(t^2-1)^2}\mathrm{d}t\right| = \int\frac{t^2-1}{3t^2+2}\cdot t\cdot\frac{-10t}{(t^2-1)^2}\mathrm{d}t =$$
$$= \int\frac{-10t^2}{(t^2-1)(3t^2+2}\mathrm{d}t$$

Parciální zlomky:

$$\frac{-10t^2}{(t^2-1)(3t^2+2)} = \frac{-10t^2}{(t-1)(t+1)(3t^2+2)} = \frac{A}{t-1}+\frac{B}{t+1}+\frac{Ct+D}{3t^2+2}$$
$$-10t^2 = (At+A)(3t^2+2)+(Bt-B)(3t^2+2)+(Ct+D)(t^2-1)$$
$$-10t^2 = 3At^3+2At+3At^2+2A+3Bt^3+2Bt-3Bt^2-2B+Ct^3-Ct+Dt^2-D$$

Porovnáním koeficientů dostáváme:

$$0=3A+3B+C$$
$$-10=3(A-B)+D$$
$$0=2A+2B-C$$
$$0=2(A-B)-D$$
$$(2,4) \implies D=-10-3(A-B)=2(A-B) \implies A-B=-2 \implies A=B-2, D=-4$$
$$\overset{1,3}{\implies} 0=6B-6+C=4B-4-C \implies C=0, B=1$$

Řešení soustavy: $A = -1, B = 1, C = 0, D = -4.$

$$\int \frac{-10t^2}{(t^2-1)(3t^2+2}\mathrm{d}t \overset{\text{lin.}}{=} -\int \frac{1}{t-1}\mathrm{d}t + \int \frac{1}{t+1}\mathrm{d}t - 4\int \frac{1}{3t^2+2}\mathrm{d}t =$$
$$\overset{\text{c}}{=} -\log|t-1| + \log|t+1| - \frac{2\sqrt{2}}{\sqrt{3}}\arctan\sqrt{\frac{3}{2}}t =$$
$$= -\log\left|\sqrt{\frac{x+2}{x-3}}-1\right| + \log\left|\sqrt{\frac{x+2}{x-3}}+1\right| - \frac{2\sqrt{2}}{\sqrt{3}}\arctan\left(\sqrt{\frac{3}{2}}\sqrt{\frac{x+2}{x-3}}\right)$$

Poslední integrály jsme jśpočetli jako již mnohokrát dříve - substitucí a úpravou výrazu...

Příklad 3 (h) $\int \frac{1-\sqrt{x+1}}{1-\sqrt[3]{x+1}}\mathrm{d}x$

$$\int \frac{1-\sqrt{x+1}}{1-\sqrt[3]{x+1}}\mathrm{d}x = \left|t = \sqrt[6]{x+1}, t^3 = \sqrt{x+1}, t^2 = \sqrt[3]{x+1}, x = t^6 - 1 \implies \mathrm{d}x = 6t^5\mathrm{d}t\right| =$$
$$= \int \frac{1-t^3}{1-t^2}6t^5\mathrm{d}t = \int \frac{6t^5(t-1)(t^2+t+1)}{(t+1)(t-1)}\mathrm{d}t = \int \frac{6t^7+6t^6+6t^5}{t+1}\mathrm{d}t =$$
$$= \int 6t^6 + 6t^4 - 6t^3 + 6t^2 - 6t + 6 - \frac{6}{t+1}\mathrm{d}t \overset{\text{c}}{=} \frac{6}{7}t^7 + \frac{6}{5}t^5 - \frac{6}{4}t^4 + \frac{6}{3}t^3 - \frac{6}{2}t^2 + 6t - 6\log|t+1| =$$
$$= \frac{6}{7}\sqrt[6]{(x+1)^7} + \frac{6}{5}\sqrt[6]{(x+1)^5} - \frac{3}{2}\sqrt[6]{(x+1)^4} + 2\sqrt[6]{(x+1)^3} - 3\sqrt[6]{(x+1)^2} + 6\sqrt[6]{x+1} -$$
$$- 6\log(\sqrt[6]{x+1}+1)$$

Příklad 3 (i) $\int \frac{1}{x(\log^3 x - 1)}\mathrm{d}x$

$$\int \frac{1}{x(\log^3 x - 1)}\mathrm{d}x = \left|y = \log x, \mathrm{d}y = \frac{1}{x}\mathrm{d}x\right| = \int \frac{1}{y^3-1}\mathrm{d}y$$

Parciální zlomky:

$$\frac{1}{y^3-1} = \frac{1}{(y-1)(y^2+y+1)} = \frac{A}{y-1} + \frac{By+C}{y^2+y+1}$$
$$1 = Ay^2 + Ay + A + By^2 - By + Cy - C$$

Porovnáním koeficientů dostáváme:

$$0 = A + B$$
$$0 = A - B + C$$
$$1 = A - C$$

Řešení soustavy: $A = \frac{1}{3}, B = -\frac{1}{3}, C = -\frac{2}{3}.$

$$\int \frac{1}{y^3-1}\mathrm{d}y \overset{\text{lin.}}{=} \frac{1}{3}\int \frac{1}{y-1}\mathrm{d}y - \frac{1}{3}\int \frac{y+2}{y^2+y+1}\mathrm{d}y = \left|u = y^2+y+1, \mathrm{d}u = (2y+1)\mathrm{d}y\right| =$$
$$= \frac{1}{3}\log|y-1| - \frac{1}{6}\int \frac{2y+4-3+3}{y^2+y+1}\mathrm{d}y = \frac{1}{3}\log|y-1| - \frac{1}{6}\int \frac{1}{u}\mathrm{d}u - \frac{1}{2}\int \frac{1}{y^2+y+1}\mathrm{d}y =$$
$$= \frac{1}{3}\log|y-1| - \frac{1}{6}\log|u| - \frac{1}{2}\int \frac{1}{\left(y+\frac{1}{2}\right)^2+\frac{3}{4}}\mathrm{d}y = \frac{1}{3}\log|y-1| - \frac{1}{6}\log|u| - \frac{2}{3}\int \frac{1}{\left(\frac{y+\frac{1}{2}}{\sqrt{\frac{3}{4}}}\right)^2+1}\mathrm{d}y =$$
$$\overset{\text{c}}{=} \frac{1}{3}\log|y-1| - \frac{1}{6}\log|u| - \frac{2}{3}\sqrt{\frac{3}{4}}\arctan\left(\frac{y+\frac{1}{2}}{\sqrt{\frac{3}{4}}}\right) =$$
$$= \frac{1}{3}\log|y-1| - \frac{1}{6}\log|y^2+y+1| - \frac{1}{\sqrt{3}}\arctan\left(\frac{1}{\sqrt{3}}\left(y+\frac{1}{2}\right)\right) =$$
$$= \frac{1}{3}\log|\log x-1| - \frac{1}{6}\log|\log^2 x+\log x+1| - \frac{1}{\sqrt{3}}\arctan\left(\frac{2}{\sqrt{3}}\left(\log x+\frac{1}{2}\right)\right)$$

Příklad 3 (j) $\int \sqrt{\frac{x-1}{x+2}}\mathrm{d}x$

Provedeme substituci: $t = \sqrt{\frac{x-1}{x+2}}$. Odvoďme $x, \mathrm{d}x$.

$$t^2 = \frac{x-1}{x+2} \implies t^2(x+2) = x-1 \implies x(t^2-1) = -1-2t^2 \implies x = \frac{-2t^2-1}{t^2-1}$$

$$x = \frac{-2t^2-1}{t^2-1} \implies \mathrm{d}x = \frac{-4t(t^2-1)-(-2t^2-1)2t}{(t^2-1)^2}\mathrm{d}t = \frac{-4t^3+4t+4t^3+2t}{(t^2-1)^2}\mathrm{d}t = \frac{6t}{(t^2-1)^2}\mathrm{d}t$$

$$\int \sqrt{\frac{x-1}{x+2}}\mathrm{d}x = \left|t = \sqrt{\frac{x-1}{x+2}}, \mathrm{d}x = \frac{6t}{(t^2-1)^2}\mathrm{d}t\right| = \int \frac{6t^2}{(t^2-1)^2}\mathrm{d}t$$

Paricální zlomky:

$$\frac{6t^2}{(t^2-1)^2} = \frac{6t^2}{(t-1)^2(t+1)^2} = \frac{A}{t-1} + \frac{B}{(t-1)^2} + \frac{C}{t+1} + \frac{D}{(t+1)^2}$$
$$6t^2 = (At-A+B)(t^2+2t+1) + (Ct+C+D)(t^2-2t+1)$$
$$6t^2 = At^3+2At^2+At-At^2-2At-A+Bt^2+2Bt+B+Ct^3-2Ct^2+Ct+Ct^2-2Ct+$$
$$+C+Dt^2-2Dt+D$$

Porovnáním koeficientů dostáváme:

$$0 = A+C$$
$$6 = A+B-C+D$$
$$0 = -A+2B-C-2D$$
$$0 = -A+B+C+D$$

Vyřešením soustavy dostaneme: $A = \frac{3}{2}, B = \frac{3}{2}, C = -\frac{3}{2}, D = \frac{3}{2}$.

$$\int \frac{6t^2}{(t^2-1)^2}\mathrm{d}t \overset{\text{lin.}}{=} \frac{3}{2}\int \frac{1}{t-1}\mathrm{d}t + \frac{3}{2}\int \frac{1}{(t-1)^2}\mathrm{d}t - \frac{3}{2}\int \frac{1}{t+1}\mathrm{d}t + \frac{3}{2}\int \frac{1}{(t+1)^2}\mathrm{d}t =$$
$$\overset{c}{=} \frac{3}{2}\log|t-1| - \frac{3}{2}\frac{1}{t+1} - \frac{3}{2}\log|t+1| - \frac{3}{2}\frac{1}{t+1} =$$
$$= \frac{3}{2}\log\left|\sqrt{\frac{x-1}{x+2}}-1\right| - \frac{3}{2}\frac{1}{\sqrt{\frac{x-1}{x+2}}+1} - \frac{3}{2}\log\left|\sqrt{\frac{x-1}{x+2}}+1\right| - \frac{3}{2}\frac{1}{\sqrt{\frac{x-1}{x+2}}+1}$$

Příklad 3 (k) $\int \frac{1}{x}\sqrt{x^2-2x}\mathrm{d}x$

Platí, že $x^2-2x = x(x-2)$. Provedeme substituci $t = \sqrt{\frac{x}{x-2}}$ (dle nápovědy z teorie k 11. cvičení).

$$t^2 = \frac{x}{x-2} \implies t^2x - 2t^2 = x \implies x = \frac{2t^2}{t^2-1} \implies \mathrm{d}x = \frac{-4t}{(t^2-1)^2}$$

$$\int \frac{1}{x}\sqrt{x^2-2x}\mathrm{d}x = \left|t = \sqrt{\frac{x}{x-2}}, \mathrm{d}x = \frac{-4t}{(t^2-1)^2}\right| = \int \frac{t^2-1}{2t^2}\sqrt{\left(\frac{2t^2}{t^2-1}\right)^2 - 2\frac{2t^2}{t^2-1}}\frac{-4t}{(t^2-1)^2}\mathrm{d}t =$$
$$= \int \frac{-2}{t(t^2-1)}\sqrt{\frac{4t^4-4t^2(t^2-1)}{(t^2-1)^2}}\mathrm{d}t = \int \frac{4t}{t(t^2-1)^2}\mathrm{d}t = \int -\frac{1}{t+1} - \frac{1}{(t+1)^2} + \frac{1}{t-1} - \frac{1}{(t-1)^2}\mathrm{d}t =$$
$$\overset{c}{=} -\log|t+1| + \frac{1}{t+1} + \log|t-1| + \frac{1}{t-1} =$$
$$= -\log\left|\sqrt{\frac{x}{x-2}}+1\right| + \frac{1}{\sqrt{\frac{x}{x-2}}+1} + \log\left|\sqrt{\frac{x}{x-2}}-1\right| + \frac{1}{\sqrt{\frac{x}{x-2}}-1}$$

Příklad 3 (l) $\int \frac{1}{x(1+2\sqrt{x}+\sqrt[3]{x})}\mathrm{d}x$

$$\int \frac{1}{x(1+2\sqrt{x}+\sqrt[3]{x})}\mathrm{d}x = \left|t = \sqrt[6]{x}, t^3 = \sqrt{x}, t^2 = \sqrt[3]{x}, x = t^6 \implies \mathrm{d}x = 6t^5\mathrm{d}t\right| =$$
$$\int \frac{1}{t^6\left(1+2t^3+t^2\right)}6t^5\mathrm{d}t = \int \frac{6}{t(2t^3+t^2+1)}\mathrm{d}t$$

Polynom $2t^3+t^2+1$ má zřejmě kořen -1. Z toho plyne následující.

$$\frac{6}{t(2t^3+t^2+1)} = \frac{6}{t(t+1)(2t^2-t+1)} = \frac{A}{t} + \frac{B}{t+1} + \frac{Ct+D}{2t^2-t+1}$$
$$6 = (At+A)(2t^2-t+1) + Bt(2t^2-t+1) + (Ct+D)(t^2+t)$$
$$6 = 2At^3 - At^2 + At + 2At^2 - At + A + 2Bt^3 - Bt^2 + Bt + Ct^3 + Ct^2 + Dt^2 + Dt$$

Porovnáním koeficientů dostáváme:

$$0 = 2A + 2B + C \overset{4}{\implies} -12 - 2B = C$$
$$0 = A - B + C + D \overset{1,3,4}{\implies} 0 = 6 - B - 2B - 12 - B \implies -4B = 6 \implies B = -\frac{3}{2}$$
$$0 = B + D \implies D = -B$$
$$6 = A$$

Řešení: $A = 6, B = -\frac{3}{2}, C = -9, D = \frac{3}{2}$

$$\int \frac{6}{t(2t^3+t^2+1)}\mathrm{d}t \overset{\text{lin.}}{=} 6\int \frac{1}{t}\mathrm{d}t - \frac{3}{2}\int \frac{1}{t+1}\mathrm{d}t + \int \frac{-9t+\frac{3}{2}}{2t^2-t+1}\mathrm{d}t = \left|y = 2t^2 - t + 1, \mathrm{d}y = (4t-1)\mathrm{d}t\right| =$$

$$= 6\log|t| - \frac{3}{2}\log|t+1| - \frac{9}{4}\int \frac{4t - \frac{2}{3} - \frac{1}{3} + \frac{1}{3}}{2t^2-t+1}\mathrm{d}t =$$

$$= 6\log|t| - \frac{3}{2}\log|t+1| - \frac{9}{4}\int \frac{1}{y}\mathrm{d}y - \frac{9}{4}\cdot\frac{1}{3}\int \frac{1}{2t^2-t+1}\mathrm{d}t =$$

$$= 6\log|t| - \frac{3}{2}\log|t+1| - \frac{9}{4}\log|y| - \frac{3}{8}\int \frac{1}{t^2-\frac{1}{2}t+\frac{1}{2}}\mathrm{d}t =$$

$$= 6\log|t| - \frac{3}{2}\log|t+1| - \frac{9}{4}\log|y| - \frac{3}{8}\int \frac{1}{\left(t-\frac{1}{4}\right)^2+\frac{7}{16}}\mathrm{d}t =$$

$$= 6\log|t| - \frac{3}{2}\log|t+1| - \frac{9}{4}\log|y| - \frac{3}{8}\cdot\frac{16}{7}\int \frac{1}{\left(\frac{t-\frac{1}{4}}{\sqrt{\frac{7}{16}}}\right)^2+1}\mathrm{d}t = \left|z = \frac{4}{\sqrt{7}}\left(t-\frac{1}{4}\right), \mathrm{d}z = \frac{4}{\sqrt{7}}\mathrm{d}t\right| =$$

$$= 6\log|t| - \frac{3}{2}\log|t+1| - \frac{9}{4}\log|y| - \frac{6}{7}\cdot\frac{\sqrt{7}}{4}\int \frac{1}{z^2+1}\mathrm{d}z =$$

$$\overset{\text{c}}{=} 6\log|t| - \frac{3}{2}\log|t+1| - \frac{9}{4}\log|y| - \frac{3}{2\sqrt{7}}\arctan z =$$

$$= 6\log|t| - \frac{3}{2}\log|t+1| - \frac{9}{4}\log|2t^2-t+1| - \frac{3}{2\sqrt{7}}\arctan\frac{4t-1}{\sqrt{7}} =$$

$$= 6\log\sqrt[6]{x} - \frac{3}{2}\log\left(\sqrt[6]{x}+1\right) - \frac{9}{4}\log|2\sqrt[3]{x} - \sqrt[6]{x} + 1| - \frac{3}{2\sqrt{7}}\arctan\frac{4\sqrt[6]{x}-1}{\sqrt{7}}$$